9 Series of functions

9.1 Power Series

The most important series is the **geometric series**:

$$a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \sum_{n=0}^{+\infty} a\,r^n\,.$$

If $-1 < r < 1$ the geometric series converges. Moreover, we proved

$$\sum_{n=0}^{+\infty} a\,r^n = a + a\,r + a\,r^2 + a\,r^3 + \cdots + a\,r^n + \cdots = \frac{a}{1-r} \quad \text{for} \quad -1 < r < 1\,. \tag{9.1.1}$$

Replacing r by x and letting $a = 1$ we can rewrite the formula in (9.1.1) as

$$\sum_{n=0}^{+\infty} x^n = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \frac{1}{1-x} \quad \text{for} \quad -1 < x < 1\,. \tag{9.1.2}$$

The formula (9.1.2) can be viewed as a representation of the function

$$f(x) = \frac{1}{1-x}, \quad -1 < x < 1,$$

as an infinite series of powers of x: $1 = x^0, x, x^2, x^3, \ldots$:

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots = \sum_{n=0}^{+\infty} x^n \quad \text{for} \quad -1 < x < 1\,.$$

You will agree that the (non-negative) integer powers of x are very simple functions. Therefore, it is natural to explore the following question:

Q1: Which functions can be represented as infinite series of constant multiples of (non-negative) integer powers of x?

In other words: Which functions $x \mapsto f(x)$ can be represented as

$$f(x) = a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \quad \text{for} \quad ?\ < x < ?\,.$$

The infinite series

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n \tag{9.1.3}$$

is called a **power series**.

The first question to answer about a power series is:

Q2: For which real numbers x does the power series converge?

Since we are working with the powers of x and since there is no restriction on the signs of a_n and x, we can use Theorems 8.6.6 and 8.6.7 (the ratio and root test) to determine the absolute convergence of the power series (9.1.3). To apply Theorem 8.6.6 we calculate

$$\lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|^{n+1}}{|a_n|\,|x|^n} = \lim_{n\to+\infty} \frac{|a_{n+1}|\,|x|}{|a_n|} = |x| \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|}.$$

Assume that

$$\lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = L. \tag{9.1.4}$$

If $L = 0$, then Theorem 8.6.6 implies that the series (9.1.3) converges for all real numbers x. If $L > 0$, then Theorem 8.6.6 implies that the series (9.1.3)

$$\begin{aligned} \text{converges absolutely for} \quad & |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L} \\ \text{diverges for} \quad & |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \ \text{ or } \ x > \frac{1}{L} \end{aligned}$$

If the limit in (9.1.4) does not exist, then no conclusion about the convergence or divergence can be deduced.

To apply Theorem 8.6.7 we calculate

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|\,|x|^n} = |x| \lim_{n\to+\infty} \sqrt[n]{|a_n|}\,.$$

Assume that

$$\lim_{n\to+\infty} \sqrt[n]{|a_n|} = L. \tag{9.1.5}$$

If $L = 0$, then Theorem 8.6.7 implies that the series (9.1.3) converges for all real numbers x. If $L > 0$, then Theorem 8.6.7 implies that the series (9.1.3)

$$\begin{aligned} \text{converges absolutely for} \quad & |x|\,L < 1, \quad \text{that is for} \quad -\frac{1}{L} < x < \frac{1}{L} \\ \text{diverges for} \quad & |x|\,L > 1, \quad \text{that is for} \quad x < -\frac{1}{L} \ \text{ or } \ x > \frac{1}{L} \end{aligned}$$

If the limit in (9.1.5) does not exist, then no conclusion about the convergence or divergence can be deduced.

Example 9.1.1. Consider the power series

$$\frac{1}{0!} + \frac{1}{1!}\,x + \frac{1}{2!}\,x^2 + \frac{1}{3!}\,x^3 + \cdots + \frac{1}{n!}\,x^n + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}\,x^n.$$

In this example $a_n = 1/n!$, $n = 0, 1, 2, \ldots$. We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)!}}{\frac{1}{n!}} = \lim_{n\to+\infty} \frac{1}{n+1} = 0.$$

Consequently the given power series converges absolutely for every $x \in \mathbb{R}$.

Example 9.1.2. Consider the power series

$$1+2x+3\,x^2+4\,x^3+\cdots+(n+1)\,x^n+\cdots=\sum_{n=0}^{\infty}(n+1)\,x^n.$$

Here $a_n=n+1$, $n=0,1,2,\ldots$ and we calculate

$$L=\lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to+\infty}\frac{n+2}{n+1}=1.$$

Consequently the given power series converges absolutely for every $x\in(-1,1)$. Clearly the series diverges for $x=-1$ and for $x=1$.

Example 9.1.3. Consider the power series

$$x-\frac{1}{2}\,x^2+\frac{1}{3}\,x^3-\frac{1}{4}\,x^4+\cdots+(-1)^{n+1}\frac{1}{n}\,x^n+\cdots=\sum_{n=0}^{\infty}(-1)^{n+1}\frac{1}{n}\,x^n.$$

Here $a_0=0$ and $a_n=(-1)^{n+1}1/n$, $n=1,2,\ldots$. We calculate

$$L=\lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to+\infty}\frac{\frac{1}{n+1}}{\frac{1}{n}}=\lim_{n\to+\infty}\frac{n}{n+1}=1.$$

Consequently the given power series converges absolutely for every $x\in(-1,1)$. Clearly the series diverges for $x=-1$ and converges conditionally for $x=1$.

Example 9.1.4. Consider the power series

$$1+\frac{1}{2}x+\frac{1}{2^2}\,x^2+\frac{1}{2^3}\,x^3+\cdots+\frac{1}{2^n}\,x^n+\cdots=\sum_{n=0}^{\infty}\frac{1}{2^n}\,x^n. \tag{9.1.6}$$

Here $a_n=2^{-n}, n=0,1,2,\ldots$. We calculate

$$L=\lim_{n\to+\infty}\frac{|a_{n+1}|}{|a_n|}=\lim_{n\to+\infty}\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}=\lim_{n\to+\infty}\frac{1}{2}=\frac{1}{2}.$$

Consequently the given power series converges absolutely for every $x\in(-2,2)$. Clearly the series diverges for $x=-2$ and for $x=2$.

Notice that we can actually calculate the sum of this series using the following substitution (or you can call this a trick). Substitute $u=x/2$ in (9.1.6). Then (9.1.6) becomes

$$1+u+u^2+u^3+\cdots+u^n+\cdots=\sum_{n=0}^{\infty}u^n. \tag{9.1.7}$$

We know that the sum of the series in (9.1.7) is $1/(1-u)$ for $u\in(-1,1)$, that is,

$$1+u+u^2+u^3+\cdots+u^n+\cdots=\sum_{n=0}^{\infty}u^n=\frac{1}{1-u},\quad u\in(-1,1).$$

Substituting back $u=x/2$ we get:

$$1+\frac{1}{2}x+\frac{1}{2^2}\,x^2+\frac{1}{2^3}\,x^3+\cdots+\frac{1}{2^n}\,x^n+\cdots=\sum_{n=0}^{\infty}\frac{1}{2^n}\,x^n=\frac{2}{2-x},\quad x\in(-2,2).$$

Example 9.1.5. Consider the power series

$$\frac{1}{1}x + \frac{1}{4}\,x^2 + \frac{1}{9}\,x^3 + \cdots + \frac{1}{n^2}\,x^n + \cdots = \sum_{n=1}^{\infty} \frac{1}{n^2}\,x^n.$$

We calculate

$$L = \lim_{n\to+\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to+\infty} \frac{\frac{1}{(n+1)^2}}{\frac{1}{n^2}} = \lim_{n\to+\infty} \frac{n^2}{(n+1)^2} = 1.$$

Consequently the given power series converges absolutely for every $x \in (-1,1)$. For $x = 1$ we get the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$. Therefore, for $x = 1$ the given power series converges. For $x = -1$ we get the alternating series which converges absolutely. Therefore the given power series converges absolutely on $[-1,1]$.

The following theorem answers the question **Q2** above.

Theorem 9.1.6. *Let*

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n$$

be a power series. Then one of the following three cases holds.

(A) *The power series converges absolutely for all $x \in \mathbb{R}$.*

(B) *There exists $r > 0$ such that the power series converges absolutely for all $x \in (-r,r)$ and diverges for all x such that $|x| > r$.*

(C) *The power series diverges for all $x \neq 0$. For $x = 0$ it is trivial that the power series converges.*

The set on which a power series converges is called the *interval of convergence*. The number $r > 0$ in Theorem 9.1.6 (B) is called the *radius of convergence*. In the case (A) in Theorem 9.1.6 we write $r = +\infty$. In the case (C) in Theorem 9.1.6 we write $r = 0$.

Remark 9.1.7. In the case (B) in Theorem 9.1.6 the convergence of the power series at the points $x = r$ and $x = -r$ must be determined by studying the infinite series

$$\sum_{n=0}^{+\infty} a_n\,r^n \quad \text{and} \quad \sum_{n=0}^{+\infty} a_n\,(-r)^n.$$

A review of the examples in this section shows that the interval of convergence of a power series can have any of these four forms $(-r,r)$, $(-r,r]$, $[-r,r)$ and $[-r,r]$.

9.2 Functions Represented as Power Series

The following theorem lists properties of functions defined by a power series.

Theorem 9.2.1. *Let I be the interval of convergence of the power series*

$$a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n.$$

Assume that I does not consist of a single point. Then the function f defined on I by

$$f(x) := a_0 + a_1\,x + a_2\,x^2 + a_3\,x^3 + \cdots + a_n\,x^n + \cdots = \sum_{n=0}^{+\infty} a_n\,x^n, \quad x \in I, \tag{9.2.1}$$

has the following three properties.

(a) *The function f is continuous on I.*

(b) *The function f is differentiable at all interior points of I. Moreover,*

$$f'(x) = a_1 + 2a_2\,x + 3a_3\,x^2 + \cdots + na_n\,x^{n-1} + (n+1)a_{n+1}x^n + \cdots = \sum_{n=0}^{+\infty}(n+1)a_{n+1}\,x^n,$$

for all $x \in I$ excluding the endpoints (if any) of I.

(c) *The function f has derivatives of all orders $1, 2, 3, \ldots,$ at all interior points of I. In particular*

$$f(0) = a_0,\ f'(0) = a_1,\ f''(0) = 2\,a_2,\ f'''(0) = 3\cdot 2\,a_3,\ \ldots,\ f^{(n)}(0) = n!\,a_n,\ \ldots. \tag{9.2.2}$$

(d) *If $x \in I$, then*

$$\int_0^x f(t)dt = a_0x + \frac{a_1}{2}\,x^2 + \frac{a_2}{3}\,x^3 + \cdots + \frac{a_{n-1}}{n}\,x^n + \frac{a_n}{n+1}\,x^{n+1} + \cdots = \sum_{n=1}^{+\infty}\frac{a_{n-1}}{n}\,x^n.$$

Example 9.2.2. By (9.1.2) we have

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \qquad \text{for} \quad -1 < x < 1\,. \tag{9.2.3}$$

Thus the function $f(x) = 1/(1-x)$ defined for $x \in (-1, 1)$ can be represented by a power series. Applying Theorem 9.2.1 we get

$$\frac{1}{(1-x)^2} = 1 + 2\,x + 3\,x^2 + 4\,x^3 + \cdots + n\,x^{n-1} + (n+1)\,x^n + \cdots \qquad \text{for} \quad -1 < x < 1\,.$$

Example 9.2.3. Substituting $-x$ for x in (9.2.3) we get

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + \cdots + (-1)^n x^n + \cdots \qquad \text{for} \quad -1 < x < 1\,. \tag{9.2.4}$$

Thus the function $f(x) = 1/(1+x)$ defined for $x \in (-1,1)$ can be represented by a power series. Applying Theorem 9.2.1 (d) we get

$$\ln(1+x) = \int_0^x \frac{1}{1+t}dt = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{4}x^4 + \cdots + (-1)^{n+1}\frac{1}{n}x^n + \cdots \qquad \text{for} \quad -1 < x < 1\,.$$

For $x = 1$ the above series is an alternating harmonic series which converges conditionally. Thus we found a power series representation for the function $\ln(1+x)$ on the interval $(-1,1]$. By Theorem 9.2.1 (a) this implies that the sum of the alternating harmonic series is $\ln 2$.

Example 9.2.4. Substituting x^2 for x in (9.2.4) we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \cdots + (-1)^n x^{2n} + \cdots \qquad \text{for} \quad -1 < x < 1\,. \tag{9.2.5}$$

Thus the function $f(x) = 1/(1+x^2)$ defined for $x \in (-1,1)$ can be represented by a power series. Applying Theorem 9.2.1 (d) we get

$$\arctan(x) = \int_0^x \frac{1}{1+t^2}dt = x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \cdots + (-1)^{n+1}\frac{1}{2n-1}x^{2n-1} + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

For $x = 1$ the above series is a conditionally convergent alternating series. Moreover,

$$\frac{\pi}{4} = \arctan 1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots + (-1)^{n+1}\frac{1}{2n-1} + \cdots$$

Thus we have a power series representation for the function $\arctan(x)$ on the interval $(-1,1]$.

9.3 Taylor series at 0 (Maclaurin series)

In the preceding section we found power series representations for several well known functions. It turns out that all well known functions can be represented as power series. The key step in finding the power series representation of elementary functions are formulas (9.2.2) which establish the relationship between the coefficients a_n, $n = 0,1,2,\ldots$, of a power series and the derivatives of the function f which is represented by that power series. We rewrite formulas (9.2.2) as

$$a_0 = f(0), \quad a_1 = f'(0), \quad a_2 = \frac{1}{2!}f''(0), \quad a_3 = \frac{1}{3!}f^{(3)}(0), \ \ldots, \ a_n = \frac{1}{n!}f^{(n)}(0), \ldots. \tag{9.3.1}$$

Let $a > 0$ and let f be a function defined on $(-a,a)$. Assume that f has all derivatives on $(-a,a)$. Then the series power series

$$f(0) + f'(0)x + \frac{1}{2!}f''(0)x^2 + \frac{1}{3!}f^{(3)}(0)x^3 + \cdots + \frac{1}{n!}f^{(n)}(0)x^n + \cdots = \sum_{n=0}^{+infty} \frac{1}{n!}f^{(n)}(0)x^n$$

is called *Taylor series at* 0 or *Maclaurin series* of f.

Using formulas (9.3.1) it is not difficult to calculate a Maclaurin series for a given function. The difficulties arise in proving that the function defined by such power series is identical to the given function. Fortunately this is true for all well known functions.

Example 9.3.1. Let $f(x) = e^x = \exp(x)$, $x \in \mathbb{R}$. Then $f^{(n)}(x) = e^x$ for all $n = 0, 1, 2, \ldots$. Therefore the coefficients of the Maclaurin series for the function exp are $a_n = 1/n!$ and it can be proved that for all $x \in \mathbb{R}$ we have

$$e^x = 1 + x + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \cdots + \frac{1}{n!}x^n + \cdots .$$

Example 9.3.2. Let $f(x) = \sin(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = \cos(x), \quad f''(x) = -\sin(x), \quad f^{(3)}(x) = -\cos(x), \quad f^{(4)}(x) = \sin(x).$$

Consequently,

$$f^{(2k)}(0) = 0, \qquad f^{(2k+1)}(0) = (-1)^k, \quad k = 0, 1, 2, \ldots .$$

Therefore the coefficients of the Maclaurin series for the function sin are

$$a_{2k} = 0, \qquad a_{2k+1} = (-1)^k \frac{1}{(2k+1)!}, \quad k = 0, 1, 2, \ldots .$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\sin(x) = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \frac{1}{7!}x^7 + \cdots + (-1)^k \frac{1}{(2k+1)!}x^{2k+1} + \cdots .$$

Example 9.3.3. Let $f(x) = \cos(x)$, $x \in \mathbb{R}$. Then

$$f'(x) = -\sin(x), \quad f''(x) = -\cos(x), \quad f^{(3)}(x) = \sin(x), \quad f^{(4)}(x) = \cos(x).$$

Consequently,

$$f^{(2k)}(0) = (-1)^k, \qquad f^{(2k+1)}(0) = 0, \quad k = 0, 1, 2, \ldots .$$

Therefore the coefficients of the Maclaurin series for the function cos are

$$a_{2k} = (-1)^k \frac{1}{(2k)!}, \qquad a_{2k+1} = 0, \quad k = 0, 1, 2, \ldots .$$

It can be proved that for all $x \in \mathbb{R}$ we have

$$\cos(x) = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \frac{1}{6!}x^6 + \cdots + (-1)^k \frac{1}{(2k)!}x^{2k} + \cdots .$$

Example 9.3.4 (The Binomial Series)**.** Let $\alpha \in \mathbb{R}$. Let $f(x) = (1+x)^\alpha$, $x \in (-1, 1)$. Then

$$\begin{aligned}
f'(x) &= \alpha(1+x)^{\alpha-1}, \\
f''(x) &= \alpha(\alpha-1)(1+x)^{\alpha-2}, \\
f^{(3)}(x) &= \alpha(\alpha-1)(\alpha-2)(1+x)^{\alpha-3}, \\
&\vdots \\
f^{(n)}(x) &= \alpha(\alpha-1)\cdots(\alpha-n+1)(1+x)^{\alpha-n} \\
&\vdots
\end{aligned}$$

Therefore the coefficients of the Maclaurin series for the function f are

$$a_0 = 1, \quad a_n = \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}, \quad n \in \mathbb{N}.$$

It can be proved that for all $x \in (-1,1)$ we have

$$(1+x)^\alpha = 1 + \frac{\alpha}{1!}x + \frac{\alpha(\alpha-1)}{2!}x^2 + \frac{\alpha(\alpha-1)(\alpha-2)}{3!}x^3 + \cdots + \frac{\alpha(\alpha-1)\cdots(\alpha-n+1)}{n!}x^n + \cdots .$$

This series is called *binomial series*. The reason for this name is that for $\alpha \in \mathbb{N}$ the binomial series becomes a polynomial:

$$\begin{aligned}
(1+x)^1 &= 1 + x \\
(1+x)^2 &= 1 + 2x + x^2 \\
(1+x)^3 &= 1 + 3x + 3x^2 + x^3 \\
(1+x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4 \\
(1+x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5 \\
(1+x)^6 &= 1 + 6x + 15x^2 + 20x^3 + 15x^4 + 6x^5 + x^6 \\
&\vdots \\
(1+x)^m &= \sum_{k=0}^{m} \binom{m}{k} x^k, \quad \text{were} \quad m \in \mathbb{N} \quad \text{and} \quad \binom{m}{k} = \frac{m!}{k!(m-k)!}
\end{aligned}$$

The last formula is called the *binomial theorem*. The coefficients

$$\binom{m}{k} = \frac{m!}{k!(m-k)!} = \frac{m(m-1)\cdots(m-k+1)}{k!} \quad \text{with} \quad m,k \in \mathbb{N}, \quad 0 \le k \le m,$$

are called *binomial coefficients*. With a general $\alpha \in \mathbb{R}$ and $k \in \mathbb{N}$ the coefficients

$$\binom{\alpha}{k} := \frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{k!}$$

are called *generalized binomial coefficients*. By definition $\binom{\alpha}{0} = 1$. With this notation the binomial series can be written as

$$(1+x)^\alpha = \sum_{k=0}^{+\infty} \binom{\alpha}{k} x^k \quad \text{for} \quad x \in (-1,1). \tag{9.3.2}$$

Notice that formula (9.2.3) is a special case of (9.3.2), since

$$\binom{-1}{k} = \frac{(-1)(-2)\cdots(-1-k+1)}{k!} = \frac{(-1)^k k!}{k!} = (-1)^k.$$

Notice also that differentiating (9.2.3) leads to

$$(1+x)^{-2} = 1 + \sum_{k=1}^{+\infty} (-1)^k (k+1)\, x^k \quad \text{for} \quad -1 < x < 1\,.$$

This is a binomial series with $\alpha = -2$. To verify this we calculate

$$\binom{-2}{k} = \frac{(-2)(-3)\cdots(-2-k+1)}{k!} = \frac{(-1)^k(k+1)!}{k!} = (-1)^k(k+1).$$

For $\alpha = 1/2$ the expression

$$\begin{aligned}\binom{1/2}{k} &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(\frac{1}{2}-k+1\right)}{k!}\\ &= \frac{\frac{1}{2}\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\cdots\left(-\frac{2k-3}{2}\right)}{k!}\\ &= \frac{(-1)^{k-1}1\cdot 3\cdot\ \cdots\ (2k-3)}{2^k\,k!}\end{aligned}$$

Thus

$$\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{2^2 2!}x^2 + \frac{1\cdot 3}{2^3 3!}x^3 - \frac{1\cdot 3\cdot 5}{2^4 4!}x^4 + \frac{1\cdot 3\cdot 5\cdot 7}{2^5 5!}x^5 + \cdots \quad \text{for} \quad -1 < x < 1\,.$$

Example 9.3.5. Let $f(x) = \arcsin(x)$, $x \in [-1,1]$. To calculate the Maclaurin series for arcsin we notice that

$$\frac{d}{dx}\big(\arcsin(x)\big) = \frac{1}{\sqrt{1-x^2}}, \qquad x \in (-1,1).$$

Now calculate the Maclaurin series for the last function using the binomial series with $\alpha = -1/2$. For $\alpha = -1/2$ and $k \in \mathbb{N}$, we calculate

$$\begin{aligned}\binom{-1/2}{k} &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{1}{2}-k+1\right)}{k!}\\ &= \frac{-\frac{1}{2}\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)\cdots\left(-\frac{2k-1}{2}\right)}{k!}\\ &= (-1)^k\frac{1\cdot 3\cdot\ \cdots\ \cdot(2k-1)}{2^k\,k!}\end{aligned}$$

Thus

$$\frac{1}{\sqrt{1+x}} = 1 - \frac{1}{2}x + \frac{1\cdot 3}{2^2 2!}x^2 + \frac{1\cdot 3\cdot 5}{2^3 3!}x^3 - \frac{1\cdot 3\cdot 5\cdot 7}{2^4 4!}x^4 + \cdots \quad \text{for} \quad -1 < x < 1\,,$$

that is,

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{+\infty}(-1)^k\frac{1\cdot 3\cdot\ \cdots\ \cdot(2k-1)}{2^k\,k!}\ x^k,$$

or using the notation of double factorials

$$\frac{1}{\sqrt{1+x}} = 1 + \sum_{k=1}^{+\infty}(-1)^k\frac{(2k-1)!!}{(2k)!!}\ x^k.$$

Substituting $-x^2$ instead of x in the above formula we get

$$\frac{1}{\sqrt{1-x^2}} = 1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k)!!}\, x^{2k}, \quad \text{for} \quad -1 < x < 1.$$

Since

$$\int_0^x \frac{1}{\sqrt{1-t^2}} dt = \arcsin(x),$$

integrating the last power series we get

$$\arcsin(x) = x + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}\, x^{2k+1} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)}\, x^{2k+1}, \quad \text{for} \quad -1 < x < 1$$

It is interesting to note that the above expansion holds at both endpoints $x = -1$ and $x = 1$. To prove this we need to recall Theorem 9.2.1 (a) and prove that the series

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!}$$

converges. This series converges by The Comparison Test. (**Hint:** Prove by mathematical induction that $\dfrac{(2k-1)!!}{(2k)!!} < \dfrac{1}{\sqrt[3]{k}}$ for all $k \in \mathbb{N}$.) As a consequence we obtain that

$$1 + \sum_{k=1}^{+\infty} \frac{(2k-1)!!}{(2k+1)(2k)!!} = \sum_{k=0}^{+\infty} \frac{\binom{2k}{k}}{4^k(2k+1)} = \frac{\pi}{2}.$$